PONZER

Interchange of the Order of Integration In a Double Integral

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THE INTERCHANGE
OF THE ORDER OF INTEGRATION
IN A DOUBLE INTEGRAL

BY

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ENTITLED Interchange of the Order of Integration,

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The Interchange of the Order of Integration in a Double Integral.

Definition of a Double Integral.

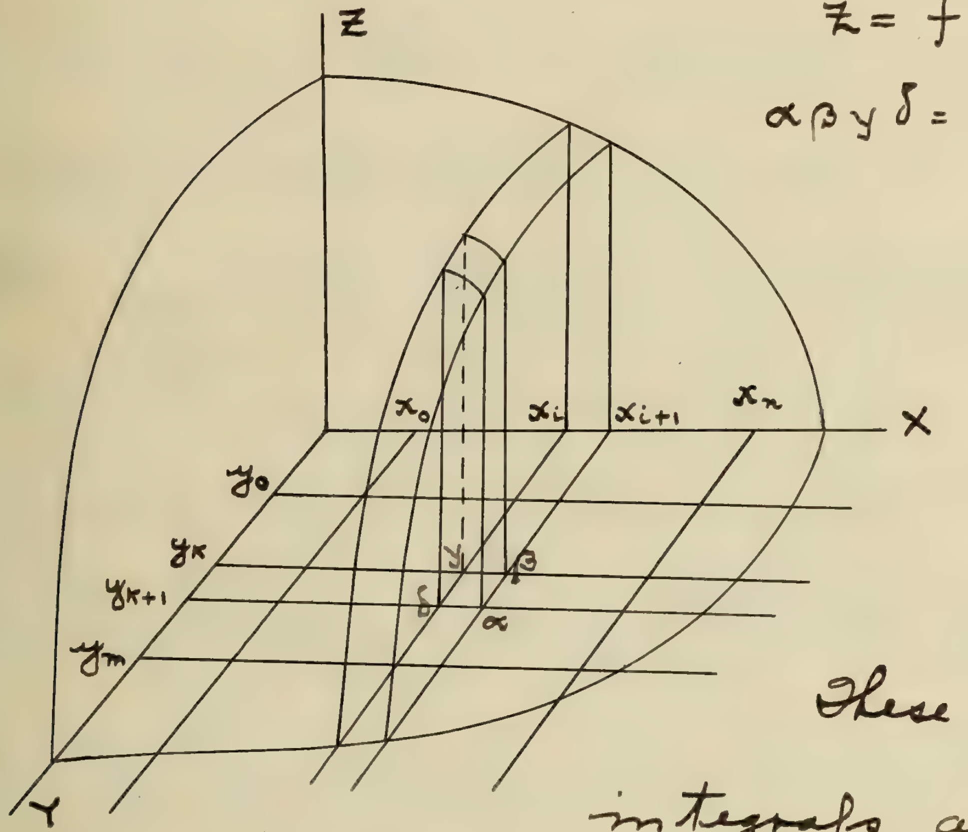
It is my purpose in this paper to discuss the conditions under which the relation

$$\int_{x_0}^{x_n} dx \int_{y_0}^{y_m} f(x,y)\,dy = \int_{y_0}^{y_m} dy \int_{x_0}^{x_n} f(x,y)\,dx$$

holds; that is, the conditions under which we may interchange the order of integration. As is well known, this is not always permissible; but in the ordinary practice of integration the right to thus interchange the order is, however, seldom questioned, and it is my purpose to investigate the problem by applying the methods

of the modern analysis to its solution.
In so doing, it will be necessary for me to define two kinds of double integrals.

In this figure
$$z = f(x, y),$$
$$\alpha\beta\gamma\delta = (x_{i+1} - x_i) \times (y_{k+1} - y_k).$$



These two integrals are the following:

① <u>Double</u> <u>Simultaneous</u> <u>Integral</u>.
We define this integral by the following identity:

*$$\int_{x_0}^{x_n}\int_{y_0}^{y_m} f(x,y)\,dx\,dy \equiv \underset{\substack{x_{i+1}-x_i \doteq 0\\ y_{k+1}-y_k \doteq 0}}{\mathrm{L}} \sum_{i=0}^{i=n-1}\sum_{k=0}^{k=m-1} f(x_i,y_k)(x_{i+1}-x_i)(y_{k+1}-y_k),$$

where $(x_1-x_0) = (x_2-x_1) = \text{-----} = (x_{i+1}-x_i) = \text{------}$

$(y_1-y_0) = (y_2-y_1) = \text{-----} = (y_{k+1}-y_k) = \text{------}$,

and where the number of terms becomes infinite as the elements $(x_{i+1}-x_i)(y_{k+1}-y_k)$ become infinitely small.

② <u>Double</u> <u>Sequential</u> <u>Integral.</u>

This integral, on the other hand, may be defined by the following identities:

$$\text{(a)} \int_{x_0}^{x_n} dx \int_{y_0}^{y_m} f(x,y)\,dy \equiv \underset{x_{i+1}-x_i \doteq 0}{\mathrm{L}}\ \underset{y_{k+1}-y_k \doteq 0}{\mathrm{L}} \sum_{i=0}^{i=n-1}\sum_{k=0}^{k=m-1} f(x_i,y_k)(x_{i+1}-x_i)(y_{k+1}-y_k)$$

$$\text{or (b)} \int_{y_0}^{y_m} dy \int_{x_0}^{x_n} f(x,y)\,dx \equiv \underset{y_{k+1}-y_k \doteq 0}{\mathrm{L}}\ \underset{x_{i+1}-x_i \doteq 0}{\mathrm{L}} \sum_{i=0}^{i=n-1}\sum_{k=0}^{k=m-1} f(x_i,y_k)(x_{i+1}-x_i)(y_{k+1}-y_k),$$

where, as before, the number of terms increases indefinitely as the size of the elements decreases towards its limit.

* Picard (Traité D'Analyse) I., p 88.

In so defining our two integrals, it will be seen that our problem is one of limits; i.e, we are basing our distinction on the manner in which our element of area $\alpha\beta\gamma\delta$, where

$$\alpha\beta\gamma\delta = (x_{i+1} - x_i)(y_{k+1} - y_k),$$

approaches its limit. If this limit is reached by a simultaneous decrease of the two magnitudes

$$(x_{i+1} - x_i),\ (y_{k+1} - y_k)$$

we have the double simultaneous integral; but, if this variation takes place in sequence, we then have the double sequential integral. The three integrals ①, ② (a), and ② (b) are generally considered to be the same, as at least no cognizance

is often taken of the fact that ② (a) and ② (b) may differ in value, or that ② (a) and ② (b) may exist and ① be meaningless.

In this discussion we define $f(x, y)$ as a finite, single-valued function of two independent real variables x and y, and consider the additional conditions which it must satisfy in order that the three integrals may have the same value, and especially when the interchange of the order of integration, as shown in ② (a) and ② (b) is permissible. If the three integrals are equal, each may be represented by a definite volume bounded by the XY plane, the surface defined by $z = f(x, y)$, and the limiting planes $x = x_0$, $x = x_n$, $y = y_0$, and $y = y_m$.

Study of Functions

Continuous in x and y together.

The simplest function which we may study in this connection is one which is continuous in x and y together at every point of the region over which our integral is taken. It has been proved* that when $f(x, y)$ is continuous in both variables together it is also continuous in each alone. Examples of this type of function are numerous and easily built up; they form by far the majority of the functions considered in the ordinary texts on the Integral Calculus. Since the functions con-

* Townsend (Begriff des Doppellimes), p. 5.

tinuous in x and y together present no discontinuities in (x, y), nor in x alone and y alone, and have a finite value at every point, the limit of the summation ① can be shown to exist. This is proved as follows:

Proof.

Let $(x_n - x_0)(y_m - y_0)$ be the region within which $f(x, y)$ is defined and therefore continuous with respect to x and y together. Let us now form the sum

$$① \quad S = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} f(x_i, y_k)(x_{i+1} - x_i)(y_{k+1} - y_k).$$

We wish to prove that this sum has a limit as the number of divisions increases indefinitely. Corresponding

to each interval where

$$(x_{i+1}-x_i)(y_{k+1}-y_k); \quad i = 0, 1, 2, 3 \text{-----}, \quad k = 0, 1, 2, 3 \text{-----}$$

we have a corresponding value of our function, which let us denote by $f(\xi_i, \eta_k)$, and which represents the value of $f(x,y)$ at any point in the region

$$(x_{i+1}-x_i)(y_{k+1}-y_k).$$

Of these values of $f(\xi_i, \eta_k)$ there will be a maximum, since $f(x,y)$ is continuous. Let this maximum value be denoted by M_{ik}.

Let us now form the sum

$$S_1 = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (M_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k).$$

This summation, when taken over the whole region, will be the upper limit of the value of the sum S, taken with the actual values of

the function in the different intervals.

Let us now form the sum

$$S_2 = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (m_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k),$$

where m_{ik} is the minimum value of $f(x,y)$ in the element of area

$$(x_{i+1}-x_i)(y_{k+1}-y_k).$$

We wish to ~~prove~~ the existence of

$$S = \lim_{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}} \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} f(\xi_i,\eta_k)(x_{i+1}-x_i)(y_{k+1}-y_k).$$

We have given

$$S_1 = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (M_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k)$$

and $$S_2 = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (m_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k)$$

$$\therefore\ S_1 - S_2 = \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (M_{ik}-m_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k)$$

where $(M_{ik}-m_{ik})$ is the oscillation of $f(x,y)$ in the region $(x_{i+1}-x_i)(y_{k+1}-y_k)$.

Since $f(x,y)$ is continuous, the limit of this oscillation is 0 as the region $(x_{i+1}-x_i)(y_{k+1}-y_k)$ approaches its limit 0.

We then have

$$\text{①} \qquad \lim_{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}} (S_1 - S_2) = 0.$$

But as we approach this limit S_1 decreases monotone and S_2 increases monotone; hence as each varies between finite limits, each of the limits

$$\text{②} \qquad \operatorname{Lim} S_1 = \lim_{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}} \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (M_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k)$$

$$\text{and } \operatorname{Lim} S_2 = \lim_{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}} \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (m_{ik})(x_{i+1}-x_i)(y_{k+1}-y_k)$$

must exist, and from the relation ① these limits must equal each other. However, for all values of the area

$$S_2 \leqq S \leqq S_1.$$

Therefore, by virtue of the equality of the limits ② we have

$$\underset{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}}{\mathrm{L}} S_1 = \underset{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}}{\mathrm{L}} S = \underset{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}}{\mathrm{L}} S_2 ,$$

which proves the existence of $\mathrm{L}S$, since we have shown the existence and equality of $\mathrm{L}S_2$ and $\mathrm{L}S_1$.

This sum S we call the definite double simultaneous integral of $f(x,y)$ over the region $(x_n - x_0)(y_m - y_0)$, and represent it symbolically by

$$\int_{y_0}^{y_m}\int_{x_0}^{x_n} f(x,y)\,dy\,dx.$$

We must now prove that the limit S of the summation is the same whatever the law of subdivision of the region $(x_n - x_0)(y_m - y_0)$ may be.

Proof.

Let S' be the sum obtained by the division into sub-regions by a second law of division. Whatever the law of division into elementary areas may be, we will insert enough points of the second division so that each elementary area obtained by the first law of subdivision will contain at least one point of division according to the second law. We can do this without any loss of generality.

Let us now consider the general term in the first summation, given by

$$f(\xi_i, \eta_k)(x_{i+1} - x_i)(y_{k+1} - y_k)$$

Corresponding to this term in our second division, we have a term of

the form

$$f(\xi'_i, \eta'_k)(x'_{i+1} - x'_i)(y'_{k+1} - y'_k).$$

Now, since the area

$$(x_{i+1} - x_i)(y_{k+1} - y_k)$$

is made up of partial areas, when the summation of the new areas is made, it will coincide with the first in the limit.

The value $f(\xi'_i, \eta'_k)$ differs from the value $f(\xi_i, \eta_k)$ by a quantity less than ε, where ε represents the oscillation discussed above of the function within the sub-region

$$(x_{i+1} - x_i)(y_{k+1} - y_k).$$

We have then that the two parts obtained in the summations S' and S over the area $\sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (x_{i+1} - x_i)(y_{k+1} - y_k)$ will differ by a quantity less than

$$\varepsilon \sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (x_{i+1}-x_i)(y_{k+1}-y_k),$$

which difference will approach 0 as a limit as the number of divisions increases indefinitely, since $f(x,y)$ is continuous within the given region. We have then

$$S = S';$$

that is, the value of the definite integral is independent of the manner in which the given region is sub-divided.

Stolz has proved* that the area over which we integrate need not be a rectangle; nor need the divisions of the area be rectangles. In fact, this area may be of any shape whatever, and the divisions may all

* Stolz (Grundzüge) III., p. 31.

be of different shapes and sizes. However, the limit of the sum of the elements of area must differ from the whole area by a quantity $< \epsilon$, where $\epsilon \doteq 0$ as a limit.

Summing up; when $f(x, y)$ is continuous in x and y together the double simultaneous integral over an area always exists. It will now be necessary for us to prove that when the double simultaneous integral over a region exists the double sequential integrals integrals exist, are equal to each other and to the double simultaneous integral over the same region. The proof of this theorem was first given by P. du Bois-Reymond.* Later.

* Journ. f. Math. 94, p 277 (1883.)

in 1885, Harnack* proved the same, making use of the upper and lower integrals and the following reasoning: The upper integral by double integration cannot be larger than the upper simultaneous integral; while the lower sequential integral cannot be smaller than the lower simultaneous integral; and, having these given, the law follows that the double simultaneous integral and double sequential integrals over a region are the same in value, and the former is evaluated through the aid of the latter. Arzelà** has also

* Harnack Diff & Int. Rechnung Vol 2. p 282.

** Mem. dell'Ist. di Bologna 5 II, p 133.

proved the same theorem. But of all the proofs of this theorem that of C. Jordan* is perhaps the earliest given, and the one most widely quoted at this time. This theorem being proved we can say here that when the double simultaneous integral over a region exists, the double sequential integrals over the same region will exist; will be equal to each other, and hence we may then always interchange the order of integration.

<u>Study of Functions</u>

<u>having a single finite (x, y) discontinuity.</u>

When $f(x, y)$ has a single finite (x, y)

* Jordan Cours D'Analyse Vol. 1, p 42.

discontinuity we can prove the existence of the double simultaneous integral as follows:

Let $(x_{i_1+1} - x_{i_1})(y_{k_1+1} - y_{k_1})$

be the element of area in which the (x, y) discontinuity occurs. Omitting from the summation the term involving this element, and adding it separately we have

$$S = \sum_{i=0}^{i=n-1}{}' \sum_{k=0}^{k=m-1}{}' f(\xi_i, \eta_k)(x_{i+1} - x_i)(y_{k+1} - y_k)$$

$$+ f(\xi_{i_1}, \eta_{k_1})(x_{i_1+1} - x_{i_1})(y_{k_1+1} - y_{k_1}),$$

where $\Sigma'\Sigma'$ includes all the terms except that involving the (x, y) discontinuity. We may make the element of area involving the discontinuity smaller than an arbi-

trarily small positive number ε; that is,

$$(x_{i_1+1}-x_{i_1})(y_{k_1+1}-y_{k_1})<\varepsilon$$

by taking the number of terms in the summation sufficiently large. Now, since $f(x,y)$ is always finite, and the (x,y) discontinuity within

$$(x_{i_1+1}-x_{i_1})(y_{k_1+1}-y_{k_1})$$

is finite, we can make the product

$$f(\xi_{i_1},\eta_{k_1})\,(x_{i_1+1}-x_{i_1})(y_{k_1+1}-y_{k_1})$$

as small as we please by taking the number of terms sufficiently large. In the limit, therefore, this product approaches zero, and we have

$$\lim_{\substack{x_{i+1}-x_i \doteq 0 \\ y_{k+1}-y_k \doteq 0}} \Sigma'\Sigma' = \int_{x_0}^{x_n}\int_{y_0}^{y_m} f(x,y)\,dx\,dy.$$

We have then that when $f(x, y)$ has a single finite (x, y) discontinuity the double simultaneous integral over a region exists; and, making use of the theorem given above*, we know then that the double sequential integrals over the same region exist, and are equal. For this class of functions we may then always interchange the order of integration.

Example I.

$$f(x, y) \equiv \frac{xy}{x^2+y^2}.$$

As an illustration of a function presenting a single finite (x, y) discontinuity we give

$$f(x, y) \equiv \frac{xy}{x^2+y^2}; \quad f(0, 0) = 0.$$

* Page 15.

This function has an (x, y) discontinuity at $x = y = 0$, for if we put $x = my$, we obtain

$$\underset{x \doteq 0}{L} \frac{xy}{x^2+y^2} = \frac{m}{1+m^2},$$ and this expression may have all values between $+\frac{1}{2}$ and $-\frac{1}{2}$ when m takes all values between $+\infty$ and $-\infty$. This function is, however, continuous in x alone and y alone, for

$$\underset{x \doteq 0}{L}\ \underset{y \doteq 0}{L} \frac{xy}{x^2+y^2} = \underset{y \doteq 0}{L}\ \underset{x \doteq 0}{L} \frac{xy}{x^2+y^2} = f(0, 0) = 0.$$

As shown above, the (x, y) discontinuity is finite; and hence, since it is finite, it follows that the double simultaneous integral

$$\int_0^1 \int_0^1 \frac{xy}{x^2+y^2}\, dx\, dy$$

exists, and is evaluated through the

aid of the double sequential integrals. If we now perform these integrations, first with respect to y, and then with respect to x, we shall obtain the following results:

$$\int_0^1 dx \int_0^1 \frac{xy}{x^2+y^2}\,dy = \int_0^1 \left[\frac{x}{2}\log(1+x^2) - \frac{x}{2}\log x^2\right]dx$$

$$= \left[\frac{x}{4}\log(1+x^2) - \frac{x^2}{4} + \frac{1}{4}\log(1+x^2) - \frac{1}{4}x^2\log x^2 + \frac{1}{4}x^2\right]_0^1$$

$$= \frac{1}{4}(2\log 2) = \frac{1}{2}\log 2.$$

If we now interchange the order of our integration we have

$$\int_0^1 dy \int_0^1 \frac{xy}{x^2+y^2}\,dx = \int_0^1 \left[\frac{y}{2}\log(1+y^2) - \frac{y}{2}\log y^2\right]dy$$

$$= \left[\frac{y}{4}\log(1+y^2) - \frac{y^2}{4} + \frac{1}{4}\log(1+y^2) - \frac{1}{4}y^2\log y^2 + \frac{1}{4}y^2\right]_0^1$$

$$= \frac{1}{4}(2\log 2) = \frac{1}{2}\log 2.$$

From this example, it will be seen

that when $f(x,y)$ has one finite (x,y) discontinuity, but is continuous in x alone and y alone the double simultaneous and sequential integrals exist, and that the value of the latter is independent of the order of integration.

Example II.

$$f(x,y) \equiv \frac{x^2}{x^2+y^2}.$$

As another illustration of a function having a single finite (x,y) discontinuity, we give

$$f(x,y) \equiv \frac{x^2}{x^2+y^2}; \text{ where } f(0,0)=0.$$

This function has an (x,y) discontinuity at the origin; for if we put

$y = mx$ in the above

function we have

$$\underset{x \doteq 0}{L} \frac{x^2}{x^2+y^2} = \frac{1}{1+m^2},$$

which expression may have all values from 1 to 0 when m has all values from $+\infty$ to $-\infty$. This function, then, has an (x, y) discontinuity at the origin but this discontinuity is finite. But, in this function, as in the case of the function considered above, we have sequential limits; in $f(x,y) \equiv \frac{xy}{x^2+y^2}$ we found these equal to each other and equal to $f(0,0) = 0$; but here we have

$$\underset{x \doteq 0}{L} \; \underset{y \doteq 0}{L} \frac{x^2}{x^2+y^2} = 1$$

but $\underset{y \doteq 0}{L} \; \underset{x \doteq 0}{L} \frac{x^2}{x^2+y^2} = 0.$

Since these two sequential limits are

different, we have that the function is not continuous in x alone and y alone; for, if it were, we would obtain the same value for our function whatever the approach to the origin might be.

Note. By the term approach we here mean that path which is taken in evaluating either double sequential integral.

Let us now see how the fact that

$f(x, y) \equiv \frac{x^2}{x^2+y^2}$ is not continuous in x alone and y alone will affect the value of the integral

$$\int_0^1 \int_0^1 \frac{x^2}{x^2+y^2}\, dy\, dx.$$

From the theory of the double simultaneous integrals, we know that a definite value exists for this integral, and that its value is given by the double sequential integrals. Evaluating the same through the aid of the latter

we have

$$\int_0^1 dx \int_0^1 \frac{x^2}{x^2+y^2}\,dy = \int_0^1 \left[x \tan^{-1}\frac{1}{x}\right] dx$$

$$= \frac{1}{2}\left[x^2 \tan^{-1}\frac{1}{x} + x - \tan^{-1} x\right]_0^1 = \frac{1}{2}.$$

If we now interchange the order of integration, we have

$$\int_0^1 dy \int_0^1 \frac{x^2}{x^2+y^2}\,dx = \int_0^1 \left[1 - y \tan^{-1}\frac{1}{y}\right] dy$$

$$= \left[y - \frac{y^2}{2}\tan^{-1}\frac{1}{y} - \frac{y}{2} + \frac{1}{2}\tan^{-1} y\right]_0^1 = \frac{1}{2}.$$

It will be seen that the values of the double sequential integrals over the area under consideration are the same whichever order of integration is taken.

From the study of this function, we may further conclude that in order that both the simultaneous and sequential integrals may exist and be equal, it is not necessary that

$f(x, y)$ be either continuous in (x, y) nor in x alone and y alone. We also note that the value of the double sequential integrals is independent of the order of integration. However, in what follows, we shall confine our study to functions which are continuous in x alone and y alone. The question arises as to what other conditions we must impose upon the function in order that the two double sequential integrals may exist, and be equal. Furthermore, does the double simultaneous integral always exist under the same circumstances?

Study of Functions having one infinite discontinuity.

As an illustration of this class of functions we give

$$f(x,y) \equiv \frac{y^2-x^2}{(y^2+x^2)^2}\text{; where } f(0,0)=0.$$

Here $\underset{\substack{x \doteq 0 \\ y \doteq 0}}{L} \frac{y^2-x^2}{(y^2+x^2)^2} = \frac{0}{0}$ does not exist;

for, if we put $y = mx$, we have

$$\underset{x \doteq 0}{L} \frac{m^2-1}{x^2(1+m^2)} = \infty,$$ and hence $f(x,y)$ has an infinite discontinuity at the point $x=y=0$.

We also have

$$\underset{y \doteq 0}{L}\ \underset{x \doteq 0}{L} \frac{y^2-x^2}{(y^2+x^2)^2} = \infty\text{; but}$$

$$\underset{x \doteq 0}{L}\ \underset{y \doteq 0}{L} \frac{y^2-x^2}{(y^2+x^2)^2} = -\infty.$$

This function is then neither continuous in (x,y), nor in x alone and y alone.

Let us now investigate the double sequential integrals of this function over the ~~region~~ $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$.

We have

$$\int_0^1 dx \int_0^1 \frac{y^2 - x^2}{(y^2+x^2)^2}\, dy = \int_0^1 \left(\frac{1}{1+x^2}\right) dx$$

$$= -\left(\tan^{-1} x\right)_0^1 = -\frac{\pi}{4}.$$

If we now interchange the order of integration, we have

$$\int_0^1 dy \int_0^1 \frac{y^2 - x^2}{(y^2+x^2)^2} = \int_0^1 \left(\frac{1}{1+y^2}\right) dy$$

$$= \left(\tan^{-1} y\right)_0^1 = \frac{\pi}{4}.$$

From this it will be seen that the two values are different, and also, that the existence of the double sequential integrals does not imply the existence of the double simultaneous integral, which in this case is meaningless.

To show that the double simultaneous integral is meaningless, let us put

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We then have $\dfrac{y^2 - x^2}{(y^2+x^2)^2} = -\dfrac{\cos 2\varphi}{r^2}$, and

hence in a quadrant round the point

$x = y = 0$

the integral $\int_0^{\frac{\pi}{2}} \int_{s_2}^{s_1} - \frac{\cos 2\varphi}{s^2} \, d\varphi \, ds$

increases logarithmically beyond any limit because of the factor

$$\int_{s_2}^{s_1} \frac{ds}{s}.$$

Let us now change the limits of the area over which we integrate.

$$\int_0^\infty dx \int_0^\infty \frac{y^2 - x^2}{(y^2 + x^2)^2} \, dy = \int_0^\infty \left[-\frac{y}{y^2 + x^2} \right]_0^\infty dx = 0.$$

Likewise, we have

$$\int_0^\infty dy \int_0^\infty \frac{y^2 - x^2}{(y^2 + x^2)^2} \, dx = \int_0^\infty \left[\frac{x}{x^2 + y^2} \right]_0^\infty dx = 0.$$

From this we may conclude that the double sequential integrals over a region need not necessarily be different when the double simultaneous integral over the same region is meaningless.

Existence of Double Simultaneous Integral.

We have seen that the double sequential integrals over a region may exist when the double simultaneous integral over the same region does not exist. We shall now investigate the conditions necessary for the existence of the double simultaneous integral when the double sequential integrals exist. The necessary and sufficient conditions for the same are given by Schoenflies* in the following theorem, which he proves:

If $f(x,y)$ in a given rectangle H has a double sequential integral taken first with respect to y and then with respect to x, and if T^{x} is the set of points of discontinuity $> \kappa$, where κ is some positive quantity > 0, for

*Schoenflies (Punktmannigfaltigkeiten), p. 197.

any line parallel to the y axis, then $f(x,y)$ always has a double simultaneous integral, and only then, when the set of points π^x, which belong to all the parallel lines is closed for every value of π.

A further study of functions presenting an infinite discontinuity shows that both the simultaneous and sequential integrals may exist when $f(x,y)$ has an infinite discontinuity. It has been proved* that in order that the double simultaneous integral may have a definite value "the order in which $f(x,y)$ becomes infinite must be lower than the second".

* Harnack (Introduction to the Calculus), p 306.

Study of Double Simultaneous Integral.

From a consideration of the foregoing examples, it will be seen that, though we have various facts concerning the possibility of the interchange of the order of integration in a double sequential integral, we have as yet no criterion for the same. However, in the examples given, we notice this fact; when $f(x, y)$ was continuous in x alone and y alone we could always interchange the order of integration. We believe that the final solution of the possibility of the interchange of the order of integration lies in the application of the properties of such a function, and

it is such a function that we shall study in the further development of the problem under consideration.

In order to arrive at a more satisfactory conclusion, it is necessary that we study the number of points at which $f(x,y)$ may have (x,y) discontinuities $> K$, where K is some positive quantity chosen at random. For, if we can prove the existence of the double simultaneous integral, we then know that the double sequential integrals exist.* So far we have proved that the double simultaneous integral exists when we have one finite (x,y) discontinuity. We may easily extend this to the case where the finite

* Pages 15 and 16.

discontinuities are more than one but still finite in number. For, let each of these points of discontinuity be enclosed by a region which in the limit it approaches zero, then the limit of the following summation:

$$\sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (D)(x_{i,+1} - x_{i,})(y_{k,+1} - y_{k,}),$$

where (D) represents the greatest discontinuity at any point, may be made to approach zero as a limit. This is because of the fact that

$$\sum_{i=0}^{i=n-1} \sum_{k=0}^{k=m-1} (x_{i,+1} - x_{i,})(y_{k,+1} - y_{k,})$$

approaches zero as a limit and (D) is finite. Here n, and hence $(n-1)$, also m, and $(m-1)$ are all finite in number. In the limit, then, we have that the value of the integral

over the region in question is the same as though the discontinuities did not exist. The proof of this theorem is in every way similar to that of the case of a function presenting but one finite (x, y) discontinuity, the proof for which was given on page 18. The only difference arises in the fact that we here have n points of discontinuity, finite but $> k$, where k is some positive quantity chosen at pleasure.

This discussion may be extended to the case where $f(x, y)$ has discontinuities $> k$ at a set of points P,* infinite in number, but closed and enumerable.

* Borel (Theorie des Fonctions), p 6.

By a closed set we mean one which contains its limiting points; that is, contains those points at which the set becomes dense. By an enumerable set of points, we mean one between the elements of which and the rational integral numbers a one-to-one correspondence may be established. Since this correspondence may be established, we may represent our enumerable set of points as follows:

$P = (\alpha_1, \alpha_2, \alpha_3, \text{------},$ where to each α there corresponds one of the rational integral numbers.

We wish now to prove that the set of points at which $f(x, y)$ has discontinuities $> k$ forms a closed set.

<u>Proof.</u> Since $f(x, y)$ has disconti-

nuities $> k$ at an enumerable set of points, we may then represent the set of points at which the discontinuities occur by $P = (\alpha_1, \alpha_2, \alpha_3, \text{--------}$. Let $(\bar{x}, \bar{y})$ be one of these points of discontinuity, and let the set be dense at α_0. Since the points $(\bar{x}, \bar{y})$ are points where the discontinuity of $f(x, y)$ is greater than k, it follows that in the neighborhood of each such point there must exist some other point (ξ, η) such that the value of the function at these two points shall differ by more than k; that is, that we have

$$\left| f(\bar{x}, \bar{y}) - f(\xi, \eta) \right| \geqq k.$$

To each $(\bar{x}, \bar{y})$ point there corresponds an (ξ, η) point arbitrarily close to it, and hence both the $(\bar{x}, \bar{y})$ points

and the (ξ, η) points are dense at α_0.
Therefore the point α_0 must itself be a point where the discontinuity of $f(x,y)$ is greater than k, for it follows from the above consideration that in every neighborhood about α_0, however small, the oscillation of the function is greater than k. Then the limit of the oscillation, that is, the discontinuity, must be greater than k. Hence the set of points P contains α_0, and is therefore a closed set.

We must now prove that the content of the enumerable and closed set P is zero.

Proof. Let us enclose the points of the set by circles as follows:
Select any arbitrarily small positive

number ε, and enclose α_1 by a circle whose area is $\frac{1}{2}\varepsilon$; α_2 by a circle whose area is $\frac{1}{(2)^2}\varepsilon$; α_3 by a circle whose area is $\frac{1}{(2)^3}\varepsilon$; &c. The total area covered by the circles can not be greater than

$$\left[\frac{1}{2}\varepsilon + \frac{1}{(2)^2}\varepsilon + \frac{1}{(2)^3}\varepsilon + \text{-----} + \frac{1}{(2)^n}\varepsilon + \text{----}\right],$$

or $$\varepsilon\left[\frac{1}{2} + \frac{1}{(2)^2} + \text{------} + \frac{1}{(2)^n} + \text{----}\right],$$

$$= \varepsilon\,(1) = \varepsilon.$$

Now, since ε may be made to approach zero as a limit, we have that the content of the closed and enumerable set P is zero.

If $f(x,y)$ has discontinuities $> \kappa$ at a set of points closed and enumerable the double simultaneous integral exists; for we may exclude from the summation the terms of the form

$$f(\xi_{i_1}, \eta_{k_1})(x_{i_1+1} - x_{i_1})(y_{k_1+1} - y_{k_1}),$$

and hence the part excluded, or added

separately, becomes less than

$$(\mathcal{D})\ \underline{\text{Lim.}} \sum\sum (x_{i_1+1}-x_{i_1})(y_{k_1+1}-y_{k_1})$$

$$=(\mathcal{D})(\varepsilon) \quad \text{where} \quad \varepsilon \doteq 0,$$

and where $(\mathcal{D})$ is the upper limit of the discontinuities. Since $(\mathcal{D})$ is finite, because our function $f(x, y)$ has only finite discontinuities, and since $\varepsilon \doteq 0$ we have that the limit of the products involving the points of discontinuity is zero, and hence the value of the whole integral, which is the limit of the summation of products over the whole region over which our integral is taken, is the same as though the set of discontinuities did not exist. We have then proved the existence of the double simultaneous

integral of $f(x, y)$ in the following cases:

① When $f(x, y)$ is continuous in (x, y) at every point of the region over which the integral is taken.

② When $f(x, y)$ has one finite (x, y) discontinuity in the region over which the integral is taken. Disc. $\omega > \kappa$.

③ When $f(x, y)$ has n finite (x, y) discontinuities in the region under consideration, n being greater than one, but finite in number. $\omega > \kappa$.

④ When $f(x, y)$ has an infinite number of (x, y) discontinuities $> \kappa$ within our area of integration but when these points form a set closed and enumerable.

Having proved that the double simultaneous integral exists in each of these cases, we then know* that the double sequential integrals over the same region exist and are equal; and hence we may always interchange the order of integration in the double sequential integrals when $f(x, y)$ satisfies the conditions given.**

* Page 15.   ** Page 40.

The question now arises: May the points where $f(x, y)$ has finite (x, y) discontinuities $> k$ form a set which is more than closed and enumerable and yet have the content zero. The further study of the integrability of $f(x, y)$, and the interchange of the order of integration in the double sequential integrals depends upon whether or not we can prove other sets of points more complex in character to have the content zero. For, in order that the double simultaneous integral may exist, it is necessary that the content of the set of points at which the finite discontinuities $> k$ occur be zero. I have been unable so far to go beyond the

closed and enumerable set, though everything seems to point in the direction of the point-wise discontinuous functions. If we can prove that the points of discontinuity of a point-wise discontinuous function have the content zero we can then say that we can always interchange the order of integration in the double sequential integrals of $f(x, y)$ over the region in question. We could further connect this point with $f(x, y)$ continuous in x alone and y alone, for $f(x, y)$ so defined cannot be more than point-wise discontinuous. It is my belief that in the study of the point-wise discontinuous functions

there lies the final solution of the possibility of the interchange of the order of integration in a double integral.